# The Wigner-Yanase entropy is not subadditive

Frank Hansen

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**Abstract**

Wigner and Yanase introduced in 1963 the Wigner-Yanase entropy defined as minus the skew information of a state with respect to a conserved observable. They proved that the Wigner-Yanase entropy is a concave function in the state and conjectured that it is subadditive with respect to the aggregation of possibly interacting subsystems. While this turned out to be true for the quantum-mechanical entropy, we negate the conjecture for the Wigner-Yanase entropy by providing a counter example.

## 1 Introduction

The Wigner-Yanase-Dyson entropy $S_p(\rho, k)$ is defined [10] by setting

$$S_p(\rho,k) = \frac{1}{2}\operatorname{Tr}[\rho^p, k][\rho^{1-p}, k] \qquad 0 < p < 1,$$

where $\rho$ is a state (or density matrix) and $k$ is a conserved (self-adjoint) observable. Since $S_p(\rho, k)$ is homogeneous in $\rho$ we may regard it as a function defined on the set of all positive definite operators on a Hilbert space $H$. If the dimension of $H$ is infinite, we usually impose the condition that $k$ is a trace class operator. Note also that $S_p(\rho, k)$ is minus the Wigner-Yanase-Dyson skew information in the state $\rho$ with respect to the (conserved) observable $k$.

Wigner and Yanase proved concavity in the state $\rho$ of the Wigner-Yanase entropy

$$S(\rho,k) = \frac{1}{2}\operatorname{Tr}[\rho^{1/2}, k]^2$$

and conjectured that also the Wigner-Yanase-Dyson entropy is concave in $\rho$ for $0 < p < 1$. The conjecture was eventually proved by Lieb [5]. The result is in line with the well known concavity of the quantum-mechanical entropy

$$S(\rho) = -\operatorname{Tr} \rho \log \rho.$$

Wigner and Yanase also conjectured [10] that the Wigner-Yanase entropy is subadditive in the following sense. Let $H_{12} = H_1 \otimes H_2$ be a tensor product of two Hilbert spaces $H_1$ and $H_2$, and let $\rho_{12}$ be a positive definite operator on $H_{12}$. In physical applications one also requires that $\operatorname{Tr} \rho_{12} = 1$. The partial trace $\rho_1 = \operatorname{Tr}_1 \rho_{12}$ is the operator on $H_1$ defined by setting

$$(\xi \mid \rho_1 \eta) = \sum_{i \in I} (\xi \otimes e_i \mid \rho_{12}(\eta \otimes e_i)) \qquad \xi, \eta \in H_1$$

where $(e_i)_{i \in I}$ is any orthonormal basis in $H_2$. We similarly define the partial trace $\rho_2 = \operatorname{Tr}_2 \rho_{12}$ on $H_2$. Let $k_1$ (respectively $k_2$) be a self-adjoint operator on $H_1$ (respectively $H_2$) and define

$$k_{12} = k_1 \otimes 1_2 + 1_1 \otimes k_2.$$

Subadditivity of the Wigner-Yanase-Dyson entropy is the condition

$$S_p(\rho_{12}, k_{12}) \le S_p(\rho_1, k_1) + S_p(\rho_2, k_2). \tag{1}$$

Wigner and Yanase proved subadditivity (1) if $\rho_{12} = \rho_1 \otimes \rho_2$ is a simple tensor of density matrices $\rho_1$ and $\rho_2$ (with equality), or if $\rho_{12}$ is a pure state [10]. Lieb also noted [5] that (1) holds if $k_1 = 0$ or $k_2 = 0$.

The notion of subadditivity for the classical entropy with respect to the aggregation of not necessarily isolated systems goes back to Gibbs [2] and were later used by Kolmogorov and Sinai. A historical account may be found in Werhl [9], cf. also Ruelle [8, Proposition 7.2.6]. It was therefore natural to expect the same properties of the quantum-mechanical entropy. Subadditivity of the quantum-mechanical entropy was proved by Landford and Robinson [4] who cited earlier partial results by Delbrück and Molière, and R. Jost, cf. also Araki and Lieb [1], while strong subadditivity were conjectured by Landford and Robinson [4] and proved by Lieb and Ruskai [6], cf. also [7].

The notion of strong subadditivity for the Wigner-Yanase-Dyson entropy is defined in the following way. Let $H_{123} = H_1 \otimes H_2 \otimes H_3$ be a tensor product of three Hilbert spaces $H_1, H_2$ and $H_3$ and let $\rho_{123}$ be a positive definite operator on $H_{123}$. We consider the partial traces

$$\rho_2 = \operatorname{Tr}_2 \rho_{123}, \quad \rho_{12} = \operatorname{Tr}_{12} \rho_{123}, \quad \rho_{23} = \operatorname{Tr}_{23} \rho_{123}$$

and define

$$k_{123} = k_1 \otimes 1_2 \otimes 1_3 + 1_1 \otimes k_2 \otimes 1_3 + 1_1 \otimes 1_2 \otimes k_3,$$

where $k_1, k_2$ and $k_3$ are self-adjoint operators on $H_1, H_2$ and $H_3$ respectively. Strong subadditivity of the Wigner-Yanase-Dyson entropy is the condition

$$S_p(\rho_{123}, k_{123}) + S_p(\rho_2, k_2) \le S_p(\rho_{12}, k_{12}) + S_p(\rho_{23}, k_{23}), \tag{2}$$

where $k_{12} = k_1 \otimes 1_2 + 1_1 \otimes k_2$ and $k_{23} = k_2 \otimes 1_3 + 1_2 \otimes k_3$.

Strong subadditivity (SSA) is for the Wigner-Yanase-Dyson entropy a stronger condition than just subadditivity (SA). This may be inferred[1] in the following way. We set $H_2 = \mathbf{C}$ and let

$$\rho_{13} = \sum_i x_i \otimes z_i, \qquad x_i \in B(H_1),\ z_i \in B(H_3)$$

be a positive semi-definite operator on the tensor product $H_1 \otimes H_3$ and set

$$\rho_{123} = \sum_i x_i \otimes 1_2 \otimes z_i.$$

We calculate the partial traces

$$\operatorname{Tr}_{13} \rho_{123} = \rho_{13}, \quad \operatorname{Tr}_{12} \rho_{123} = \operatorname{Tr}_1 \rho_{123}, \quad \operatorname{Tr}_{23} \rho_{123} = \operatorname{Tr}_3 \rho_{123}$$

and by setting $k_2 = 1_2$, we obtain $S_p(\rho_{123}, k_{123}) = S_p(\rho_{13}, k_{13})$ and

$$S_p(\rho_2, k_2) = 0, \quad S_p(\rho_{12}, k_{12}) = S_p(\rho_1, k_1), \quad S_p(\rho_{23}, k_{23}) = S_p(\rho_3, k_3).$$

Strong subadditivity thus entails

$$S_p(\rho_{13}, k_{13}) \le S_p(\rho_1, k_1) + S_p(\rho_3, k_3),$$

which is subadditivity.

## 2 The WY-entropy is not subadditive

We shall demonstrate that the Wigner-Yanase entropy is not subadditive (and therefore not strongly subadditive either).

[1] The author is indebted to E. Lieb and R. Seiringer for providing this argument.

Consider the tensor product $H_{12} = H_1 \otimes H_2$ of two Hilbert spaces, each of dimension 2, with fixed orthonormal bases $(e_1, e_2)$ in $H_1$ and $(f_1, f_2)$ in $H_2$. We introduce the positive definite operator $\rho_{12}$ on $H_{12}$ with matrix representation

$$\rho_{12} = \begin{pmatrix} 7 & 5 & 5 & 6 \\ 5 & 6 & 2 & 5 \\ 5 & 2 & 6 & 5 \\ 6 & 5 & 5 & 7 \end{pmatrix}$$

and eigenvalues $\frac{1}{2}(21 + 5\sqrt{17}), 4, 1, \frac{1}{2}(21 - 5\sqrt{17})$. The coordinates of for example $\rho_{12}(e_2 \otimes f_1)$ are then given by column no. 3 in the matrix representing $\rho_{12}$. The positive definite square root $\rho_{12}^{1/2}$ is represented by

$$\rho_{12}^{1/2} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

and have eigenvalues $\frac{1}{2}(5 + \sqrt{17}), 2, 1, \frac{1}{2}(5 - \sqrt{17})$.

We also consider the operators on $H_1$ and $H_2$ given by

$$k_1 = \begin{pmatrix} 10 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad k_2 = \begin{pmatrix} 1 & 1 \\ 1 & 10 \end{pmatrix},$$

and calculate

$$k_1 \otimes 1_2 = \begin{pmatrix} 10 & 0 & 1 & 0 \\ 0 & 10 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad 1_1 \otimes k_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 10 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 10 \end{pmatrix}$$

and

$$k_{12} = k_1 \otimes 1_2 + 1_1 \otimes k_2 = \begin{pmatrix} 11 & 1 & 1 & 0 \\ 1 & 20 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 11 \end{pmatrix}.$$

The commutator

$$[\rho_{12}^{1/2}, k_{12}] = \begin{pmatrix} 0 & 10 & -8 & 0 \\ -10 & 0 & 0 & -10 \\ 8 & 0 & 0 & 8 \\ 0 & 10 & -8 & 0 \end{pmatrix}$$

has a square

$$[\rho_{12}^{1/2}, k_{12}]^2 = \begin{pmatrix} -164 & 0 & 0 & -164 \\ 0 & -200 & 160 & 0 \\ 0 & 160 & -128 & 0 \\ -164 & 0 & 0 & -164 \end{pmatrix}$$

and therefore

$$S(\rho_{12}, k_{12}) = \frac{1}{2}\,\mathrm{Tr}[\rho_{12}^{1/2}, k_{12}]^2 = -328.$$

We calculate the partial traces

$$\rho_1 = \rho_2 = \begin{pmatrix} 13 & 10 \\ 10 & 13 \end{pmatrix}$$

with eigenvalues $(3, 23)$ and positive definite square roots

$$\rho_1^{1/2} = \rho_2^{1/2} = \begin{pmatrix} \frac{\sqrt{3}+\sqrt{23}}{2} & \frac{-\sqrt{3}+\sqrt{23}}{2} \\ \frac{-\sqrt{3}+\sqrt{23}}{2} & \frac{\sqrt{3}+\sqrt{23}}{2} \end{pmatrix}.$$

The commutators

$$[\rho_1^{1/2}, k_1] = -[\rho_2^{1/2}, k_2] = \begin{pmatrix} 0 & -\frac{9}{2}(-\sqrt{3}+\sqrt{23}) \\ \frac{9}{2}(-\sqrt{3}+\sqrt{23}) & 0 \end{pmatrix}$$

have squares

$$[\rho_1^{1/2}, k_1]^2 = [\rho_2^{1/2}, k_2]^2 = \begin{pmatrix} -\frac{81}{4}(-\sqrt{3}+\sqrt{23})^2 & 0 \\ 0 & -\frac{81}{4}(-\sqrt{3}+\sqrt{23})^2 \end{pmatrix}$$

and therefore

$$S(\rho_1, k_1) = S(\rho_2, k_2) = \frac{1}{2}\,\mathrm{Tr}[\rho_1^{1/2}, k_1]^2 = -\frac{81}{4}(-\sqrt{3}+\sqrt{23})^2.$$

In conclusion we obtain

$$\begin{aligned} S(\rho_1, k_1) + S(\rho_2, k_2) - S(\rho_{12}, k_{12}) \quad &= -\frac{81}{2}(-\sqrt{3}+\sqrt{23})^2 + 328 \\ &= -725 + 81\sqrt{69} \\ &\approx -52.1635, \end{aligned}$$

which contradicts the conjecture of subadditivity of the Wigner-Yanase entropy. We report without giving a proof that also the Wigner-Yanase-Dyson entropies, for 0<p<1, fail to be subadditive.

## 3 More general entropies

The failure of the WYD-entropies $S_p(\rho, k)$ to be subadditive is not related to the term $\rho^p$ as one might expect. In a recent paper [3] we introduced, for each regular monotone metric with Morozova-Chentsov function $c$, a so called metric adjusted skew information $I^c(\rho, k)$ which is a generalization of the Wigner-Yanase-Dyson skew information. The extreme points in the convex set of Morozova-Chentsov functions are given by the functions

$$c_\lambda(x,y) = \frac{1+\lambda}{2}\left(\frac{1}{x+\lambda y} + \frac{1}{\lambda x + y}\right) \qquad x,y>0, \tag{3}$$

where $\lambda \in [0,1]$. We introduce for each $\lambda \in (0,1]$ the $\lambda$-entropy

$$E_\lambda(\rho,k) = -I^{c_\lambda}(\rho,k) = -\operatorname{Tr}\rho k^2 + \operatorname{Tr} k\, f_\lambda(L_\rho, R_\rho)k \tag{4}$$

specified by the function

$$f_\lambda(x,y) = xy \cdot c_\lambda(x,y) \qquad x,y>0.$$

Each $\lambda$-entropy is a concave function in $\rho$, and it is additive with respect to the aggregation of isolated subsystems [3]. The $\lambda$-entropies are vanishing for commuting operators $\rho$ and $k$ and have therefore no classical counterparts. Furthermore, the Wigner-Yanase-Dyson entropy has the following representation

$$S_p(\rho,k) = \frac{p(1-p)}{2}\int_0^1 E_\lambda(\rho,k)\frac{(1+\lambda)^2}{\lambda}\,d\mu_p(\lambda) \qquad 0<p<1, \tag{5}$$

where

$$d\mu_p(\lambda) = \frac{2\sin p\pi}{\pi p(1-p)} \cdot \frac{\lambda^p + \lambda^{1-p}}{(1+\lambda)^3}\,d\lambda$$

is a probability measure on $[0,1]$ such that $\int_0^1 \lambda^{-1} d\mu(\lambda) < \infty$.

Although the $\lambda$-entropies are much simpler than the WYD-entropies, we realize from equation (5) that at least one of them must fail to be subadditive, and this applies then to all of them because of their similarities.

# References


[1] H. Araki and E. Lieb. Entropy inequalities. *Commun. Math. Phys.*, 18:160–170, 1970.

[2] J.W. Gibbs. *The collected works of J. Willard Gibbs*. 1928. p. 159.

[3] F. Hansen. Metric adjusted skew information. *arXiv: math-ph/0607049*, pages 1–18, 2006.

[4] O.E. Lanford III and D.W. Robinson. Mean entropy of states in quantum-statistical mechanics. *Journal of Mathematical Physics*, 9:1120–1125, 1968.

[5] E. Lieb. Convex trace functions and the Wigner-Yanase-Dyson conjecture. *Advances in Math.*, 11:267–288, 1973.

[6] E. Lieb and M.B. Ruskai. Proof of the strong subadditivity of quantum-mechanical entropy. *J. Math. Phys.*, 14:1938–1941, 1973.

[7] M.A. Nielsen and D. Petz. A simple proof of the strong subadditivity inequality. *Quantum Information & Computation*, 5:507–513, 2005.

[8] D. Ruelle. *Statistical Mechanics.* W.A. Benjamin, New York, 1969.

[9] A. Wehrl. General properties of entropy. *Rev. Modern Phys.*, 50:221–260, 1978.

[10] E.P. Wigner and M.M. Yanase. Information contents of distributions. *Proc. Natl. Acad. Sci. U.S.A.*, 49:910–918, 1963.

Frank Hansen: Department of Economics, University of Copenhagen, Studiestraede 6, DK-1455 Copenhagen K, Denmark.